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GENERAL LAGRANGIAN INTERPOLATION FORMULAS

by C. E. Velez

Goddard Space Flight Center
Greenbelt, Md.

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ABSTRACT

The development of high-degree interpolation polynomials which use the values of the function and its subsequent derivatives is discussed. It is shown that if data of this type are available, high-accuracy interpolation is possible under the restrictive conditions of large step-sizes and few data values.

CONTENTS

GENERAL LAGRANGIAN INTERPOLATION FORMULAS

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C. E. Velez
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INTRODUCTION

A computing system requiring an accurate numerical interpolation process may have, in addition to the function, information concerning its derivatives at some set of discrete points. For example, consider a multistep numerical integration process designed to solve an equation of the form

$$y \qquad F(x, y, y') .$$

The data concerning the function $y(x)$ available at some point during the computation could then be written as the array

$$\begin{array}{lll}
y(x_0) & y'(x_0) & y''(x_0) \\
y(x_0 + h) & y'(x_0 + h) & y''(x_0 + h) \\
y(x_0 + 2h) & y'(x_0 + 2h) & y''(x_0 + 2h) , \\
\vdots & \vdots & \vdots \\
y(x_0 + kh) & y'(x_0 + kh) & y''(x_0 + kh)
\end{array}$$

where h is the integration stepsize, and k the number of "back points" required by the process. If the interval h of integration is modified, so that values are required for y, y' and y'', at some points $x \in (x_0, x_0 + kh)$, then an approximation of these required values could be computed by applying an interpolation polynomial of the form

(a) $$P(x) \qquad \sum_{i=0}^{n} a_i(x) \, f(x_0 + ih) \qquad n \le k$$

to each column of the array independently (with appropriate coefficients), or by differentiating the polynomial approximation of y to obtain values for y' and y''. However neither of these methods uses all the information available about the function near the point x; for example, the approximation polynomial for y would disregard the values $y'(x_0 + ih)$ and $y''(x_0 + ih)$. Therefore, consider using an interpolating polynomial of the form

(b) $$P(x) = \sum_{i=0}^{n} a_i(x)\, f(x_0 + ih) + b_i(x)\, f'(x_0 + ih) + c_i(x)\, f''(x_0 + ih)$$

to obtain an approximation for y. Using such a polynomial would increase the accuracy concomitant with the use of an increased amount of information about the function near the point x. Also an accurate interpolation may be possible even in a situation where the number of values of the function (k) is insufficient, or the stepsize (h) is too large for an accurate interpolation using a polynomial of the form (a). In the following paragraphs, the development of polynomials of the form (b) will be discussed for the case in which an arbitrary number of successive derivatives are available at a set of discrete points. In addition, the applicability of such polynomials will be demonstrated numerically.

THE GENERAL INTERPOLATING POLYNOMIAL

Let f be an N- times differentiable* function over the real line and assume the following data over the interval $[x_0, x_k]$ are given:

$$f_i^{(n)} \text{ for } \begin{cases} i = 0, 1, 2, \cdots k \\ n = 0, 1, 2, \cdots N' \end{cases},$$

where $N' \le N, f_i^{(n)} = f^{(n)}(x_i)$, and $x_i \in [x_0, x_k]$. The problem to be discussed then, is the formulation of polynomials of the form

$$P^{(n)}(x) = \sum_{j=0}^{k} \sum_{i=0}^{N'} h_{ij}^{(n)}(x)\, f_j^{(i)}, \tag{1}$$

satisfying

$$P^{(n)}(x_i) = f_i^{(n)}, \quad i = 0, 1, 2, \cdots k, \tag{2}$$

with remainder $R_n(x) = f^{(n)}(x) - P^{(n)}(x)$, where $x \in [x_0, x_k]$.

*It is assumed that higher order derivatives remain bounded in the region of interest.

Some well-known examples of polynomials of this type are:

(1) $N' = 0$, the Lagrangian interpolation polynomial of degree $k - n$ with its coefficients given by

$$h_{0,j}^{(n)}(x) = \ell_j^{(n)}(x) = \frac{d^n}{dx^n}\left\{\pi' \left[(x - x_i)/(x_j - x_i)\right]\right\}, \tag{3}$$

where

$$\pi' = \prod_{\substack{i=0 \\ i \neq j}}^{k} ;$$

(2) $N' = 1$, the Hermite interpolation polynomial of degree $(2k + 1) - n$ with its coefficients given by

$$\begin{aligned} h_{0,j}^{(n)}(x) &= \frac{d^n}{dx^n}\left[\left\{1 - 2(x - x_j)\, \Sigma' \frac{1}{x_j - x_i}\right\} \ell_j^2(x)\right] \quad \text{and} \\ h_{1,j}^{(n)}(x) &= \frac{d^n}{dx^n}\left[(x - x_j)\, \ell_j^2(x)\right], \end{aligned} \tag{4}$$

where $\ell_j(x)$ is the Lagrange coefficient given by Equation 3 for $n = 0$ and

$$\Sigma' = \sum_{\substack{i=0 \\ i \neq j}}^{k} ;$$

(3) $k = 0$ and $n = 0$, the truncated Taylor expansion about the point x_0.

In the following paragraphs, a general scheme for obtaining the coefficients of polynomials of the type given in Equation 1 will be discussed, and explicit formulas of the Equations 3 and 4 types will be given for $N' = 0, 1, 2$ and $n = 0, 1, 2$.

DETERMINATION OF THE COEFFICIENTS

Since $(N' + 1)(k + 1)$ data points are given, it is seen that the coefficients of Equation 1 could be uniquely chosen so that $P^{(n)}(x)$ is a polynomial of degree $k(N' + 1) + (N' - n)$. Consider the case $n = 0$ and let $\nu = k(N' + 1) + N'$. In this case, such a polynomial could be constructed by solving the following determinantal equation for $P(x)$ (Reference 1, pp. 33-34):

$$\begin{vmatrix} P(x) & 1 & x & x^2 \cdots\cdots & x^\nu \\ f_0 & 1 & x_0 & x_0^2 \cdots\cdots & x_0^\nu \\ f_1 & 1 & x_1 & x_1^2 & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_k & 1 & x_k & x_k^2 \cdots\cdots & x_k^\nu \\ f'_0 & 0 & 1 & 2x_0 \cdots\cdots & \nu x_k^{\nu-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f'_k & \vdots & 1 & 2x_k \cdots\cdots & \vdots \\ \vdots & \vdots & 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_k^{(N')} & 0 & 0 & \cdots & \nu(\nu-1)(\nu-N')x_k^{\nu-(N'+1)} \end{vmatrix} = 0 \cdot \tag{5}$$

This technique, however, even for $N' = 1$, involves formidable computations. An alternate method which simplifies this computation considerably is suggested by Householder (Reference 2, pp. 194-195) for $N' = 1$. This technique, extended to the case of arbitrary N', is as follows: From Equations 1 and 2,

$$f_p = \sum_{j=0}^{k} \sum_{i=0}^{N'} h_{i,j}(x_p)\, f_j^{(i)} . \tag{6}$$

This relation must hold, whatever the values of $f_j^{(i)}$; in particular, we may have $f_j = \delta_{j,p}$ and $f_j^{(i)} = 0$ for $i = 1, 2, \ldots N'$ and all j, so that Equation 6 reduces to

$$f_p = 1 = h_{0,p}(x_p) ;$$

but, since p is arbitrary,

$$h_{0,j}(x_j) = 1, \quad j = 0, 1, \cdots k .$$

Also, if for some $m \neq p$, $f_m = 1$, while $f_j = 0$ for $j \neq m$, and $f_j^{(i)} = 0$ for $i = 1, 2, \ldots N'$ and all j, then

$$f_p = 0 = h_{0,m}(x_p) ;$$

i.e.,

$$h_{0,j}(x_q) = \delta_{j,q} \cdot \tag{7}$$

Moreover, if for some n, $f_m^{(n)} = 1$, while all other $f_j^{(i)} = 0$, then

$$f_p = 0 = h_{n,m}(x_p) \quad ;$$

i.e.,

$$h_{i,j}(x_q) = 0 \text{ for } i = 1, 2, \ldots N' . \tag{8}$$

Combining Equations 7 and 8 gives

$$h_{i,j}(x_q) = \delta_{0,i}\ \delta_{i,q} \quad .$$

Repeating the foregoing argument for the polynomials $P^{(n)}(x)$, $n = 1, 2, \ldots N'$, we obtain

$$h_{i,j}^{(n)}(x_q) = \delta_{i,n}\,\delta_{j,q} \quad . \tag{9}$$

Now consider the equation

$$h_{i,j}(x) = \phi_{i,j}(x)\,\ell_j^{(N'+1)}(x) \quad , \tag{10}$$

where the $\ell_j(x)$ are the Lagrange polynomials of degree k, and $\phi_{i,j}(x)$ is a polynomial of degree N'; note that $h_{i,j}(x)$ is then a polynomial of degree ν and satisfies Equation 9 for $q \neq j$. Using the fact that a polynomial of degree ν satisfying Equation 2 is unique, reduces the problem to determining $\phi_{i,j}(x)$ so that Equation 9 is satisfied for $q = j$. This can be done by determining the N' derivatives of $\phi_{i,j}$ evaluated at the point $x = x_j$ using Equations 9 and 10, and then forming the polynomial

$$\phi_{i,j}(x) = \sum_{n=0}^{N'} (x - x_j)^n \frac{\phi_{i,j}^{(n)}(x_j)}{n!} \quad . \tag{11}$$

We will then have the required polynomial

$$P(x) = \sum_{j=0}^{k} \sum_{i=0}^{N'} h_{i,j}(x)\ f_j^{(i)} \quad , \tag{12}$$

with the coefficients $h_{i,j}$ given by Equation 10. Polynomials $P^{(n)}(x)$ for $n \neq 0$ can then be found by differentiating Equation 12:

$$P^{(n)}(x) = \frac{d^n}{dx^n} P(x) = \sum_{j=0}^{k} \sum_{i=0}^{N'} h_{i,j}^{(n)}(x)\ f_j^{(i)} \quad .$$

The error term for $P(x)$ can be derived using Equation 5 and the mean value theorem precisely as indicated in Reference 1, pp. 22-24 for the case $n = 0$. This technique yields

$$R(x) = \frac{\left(\prod_{i=0}^{k} (x - x_i)\right)^{N'+1} f^{(\nu+1)}(\xi)}{(\nu + 1)!} , \tag{13}$$

where $\xi \in [x_0, x_k]$. As before, error terms for $P^{(n)}(x)$ can then be obtained by appropriate differentiation of Equation 13, noting that ξ is a function of x. Details concerning this differentiation can be found in Reference 3, pp. 66-67.

SAMPLE DERIVATION

As an illustration of the foregoing, a polynomial of the type

$$P(x) = \sum_{j=0}^{k} \sum_{i=0}^{2} h_{i,j}(x)\, f_j^{(i)}$$

will be determined. From Equation 9, we have the conditions

$$\left.\begin{array}{lll} h_{0,j}(x_q) = \delta_{j,q} & h_{1,j}(x_q) = 0 & h_{2,j}(x_q) = 0 \\ h'_{0,j}(x_q) = 0 & h'_{1,j}(x_q) = \delta_{j,q} & h'_{2,j}(x_q) = 0 \\ h''_{0,j}(x_q) = 0 & h''_{1,j}(x_q) = 0 & h''_{2,j}(x_q) = \delta_{j,q} \end{array}\right\} , \tag{14}$$

from which to determine polynomials of degree $3k + 2$ of the form

$$h_{i,j}(x) = \phi_{i,j}(x)\, \ell_j^3(x) , \tag{15}$$

where $\phi_{i,j}(x)$ are quadratic polynomials. Differentiating Equation 15, we have

$$\left.\begin{array}{l} h'_{i,j}(x) = \phi'_{i,j}(x)\, \ell_j^3(x) + 3\, \phi_{i,j}(x)\, \ell_j^2(x)\, \ell'_j(x) \\ h''_{i,j}(x) = \phi''_{i,j}(x)\, \ell_j^3(x) + 6\, \phi'_{i,j}(x)\, \ell_j^2(x)\, \ell'_j(x) \\ \qquad + \phi_{i,j}(x)\left[6\ell_j(x)\, \ell_j'^2(x) + 3\ell''_j(x)\, \ell_j^2(x)\right] \end{array}\right\} . \tag{16}$$

From Equations 14, 15, and 16 we see that

$$h_{0,j}(x_j) = \phi_{0,j}(x_j) = 1$$

$$h_{1,j}(x_j) = \phi_{1,j}(x_j) = 0$$

$$h_{2,j}(x_j) = \phi_{2,j}(x_j) = 0 \ ;$$

also

$$h'_{0,j}(x_j) = \phi'_{0,j}(x_j) + 3\ell'_j(x_j) = 0 ,$$

that is,

$$\phi'_{0,j}(x_j) = -3\ell'_j(x_j)$$

$$h'_{1,j}(x_j) = \phi'_{1,j}(x_j) = 1$$

$$h'_{2,j}(x_j) = \phi'_{2,j}(x_j) = 0 \ ;$$

also

$$h''_{0,j}(x_j) = \phi''_{0,j}(x_j) - 12\ell'^2_j(x_j) + 3\ell''_j(x_j) = 0 ,$$

that is,

$$\phi''_{0,j}(x_j) = 12\ell'^2_j(x_j) - 3\ell''_j(x_j) ,$$

$$h''_{1,j}(x_j) = \phi''_{1,j}(x_j) + 6\ell'_j(x_j) = 0 ,$$

that is,

$$\phi''_{1,j}(x_j) = -6\ell'_j(x_j)$$

$$h''_{2,j}(x_j) = \phi''_{2,j}(x_j) = 1 .$$

Hence, using Equation 11, we have

$$\phi_{0,j}(x) = 1 - 3(x - x_j)\ell'_j(x_j) + \frac{(x - x_j)^2}{2}\left[12\ell'^2_j(x_j) - 3\ell''_j(x_j)\right] ,$$

$$\phi_{1,j}(x) = (x - x_j) - 3(x - x_j)^2\ \ell'_j(x_j) ,$$

$$\phi_{2,j}(x) = \frac{(x - x_j)^2}{2} ,$$

and the required polynomial is given by

$$P(x) = \sum_{j=0}^{k} \left[\sum_{i=0}^{2} \phi_{i,j}(x)\, f_j^{(i)} \right] \ell_j^3(x) . \tag{17}$$

FORMULAS FOR N' = 0, 1, 2; n = 0, 1, 2

Case N' = 0

For $N' = 0$,

$$P(x) = \sum_{j=0}^{k} h_{0,j}(x)\, f_j \qquad \text{(Lagrange)} , \tag{18a}$$

where

$$h_{0,j}(x) = \ell_j(x) = \pi' \left[(x - x_i) / (x_j - x_i) \right] ,$$

and

$$P'(x) = \sum_{j=0}^{k} h'_{0,j}(x)\, f_j , \tag{18b}$$

where

$$h'_{0,j}(x) = \ell'_j(x) = \ell_j(x)\, \Sigma' \frac{1}{(x - x_i)} ;$$

and

$$P''(x) = \sum_{j=0}^{k} h''_{0,j}(x)\, f_j , \tag{18c}$$

where

$$h''_{0,j}(x) = \ell''_j(x) = \ell_j(x) \left\{ \left[\Sigma' \frac{1}{(x - x_i)^2} \right]^2 - \Sigma' \frac{1}{(x - x_i)^2} \right\} .$$

Case $N' = 1$

For $N' = 1$,

$$P(x) = \sum_{j=0}^{k} h_{0,j}(x)\ f_j + h_{1,j}(x)\ f_j' \qquad \text{(Hermite)}, \tag{19a}$$

where

$$h_{i,j}(x) = \phi_{i,j}(x)\ \ell_j^2(x),$$

and

$$\phi_{0,j}(x) = 1 - 2(x - x_j)\ \ell_j'(x_j),$$

$$\phi_{1,j}(x) = (x - x_j),$$

where $\ell_j'(x_j)$ is given in Equation 18b with $x = x_j$. By differentiating, we obtain

$$P'(x) = \sum_{j=0}^{k} h_{0,j}'(x)\ f_j + h_{1,j}'(x)\ f_j', \tag{19b}$$

where

$$h_{i,j}(x) = \gamma_{i,j}(x)\ \ell_j^2(x),$$

$$\gamma_{0,j}(x) = 2\left\{\sum{}' \frac{1}{(x - x_i)} - (x - x_j)\ \ell_j'(x_j)\left[2\sum{}' \frac{1}{(x - x_i)} + \frac{1}{(x - x_j)}\right]\right\},$$

$$\gamma_{1,j}(x) = 1 + 2(x - x_j) \sum{}' \frac{1}{(x - x_i)};$$

and

$$P''(x) = \sum_{j=0}^{k} h_{0,j}''(x)\ f_j + h_{1,j}''(x)\ f_j', \tag{19c}$$

where

$$h_{i,j}(x) = \gamma_{i,j}(x)\ \ell_j^2(x),$$

$$\gamma_{0,j}(x) = 2\left\{\left[2\left(\Sigma'\frac{1}{x-x_i}\right)^2 - \Sigma'\frac{1}{(x-x_i)^2}\right]\left[1 - 2(x-x_j)\,\ell_j'(x_j)\right] - 4\,\Sigma'\frac{1}{(x-x_i)}\,\ell_j'(x_j)\right\},$$

$$\gamma_{1,j}(x) = 4\,\Sigma'\frac{1}{(x-x_i)} + 2(x-x_j)\left\{2\left[\Sigma'\frac{1}{(x-x_i)^2}\right]^2 - \Sigma'\frac{1}{(x-x_i)^2}\right\}.$$

Case N′ = 2

For $N' = 2$,

$$P(x) = \sum_{j=0}^{k} h_{0,j}(x)\,f_j + h_{1,j}(x)\,f_j' + h_{2,j}(x)\,f_j'', \tag{20a}$$

where

$$h_{i,j}(x) = \phi_{i,j}(x)\,\ell_j^3(x),$$

and

$$\phi_{0,j}(x) = 1 + 6(x-x_j)^2\left\{\left[\ell_j'(x_j)\right]^2 - \frac{1}{4}\,\ell_j''(x_j)\right\} - 3(x-x_j)\,\ell_j'(x_j),$$

where $\ell_j''(x_j)$ is given by Equation 18c with $x = x_j$;

$$\phi_{1,j}(x) = (x-x_j) - 3(x-x_j)^2\,\ell_j'(x_j)$$

$$\phi_{2,j}(x) = \frac{(x-x_j)^2}{2}.$$

By differentiating, we obtain

$$P'(x) = \sum_{j=0}^{k} h_{0,j}'(x)\,f_j + h_{1,j}'(x)\,f_j' + h_{2,j}'(x)\,f_j'', \tag{20b}$$

where

$$h_{i,j}'(x) \quad \gamma_{i,j}(x)\,\ell_j^3(x)$$

and

$$\gamma_{i,j}(x) \quad 3\,\phi_{i,j}(x)\,\Sigma'\frac{1}{(x-x_i)} + \phi_{i,j}'(x),$$

where $\phi_{i,j}(x)$ is given in Equation 19a, and

$$\phi'_{0,j}(x) = 12\left(x - x_j\right)\left[\ell_j'^{\,2}\left(x_j\right) - \frac{1}{4}\ell_j''\left(x_j\right)\right] - 3\ell_j' \; x_j$$

$$\phi'_{1,j}(x) = 1 - 6\left(x - x_j\right)\ell_j'\left(x_j\right)$$

$$\phi'_{2,j}(x) = \left(x - x_j\right);$$

and

$$P''(x) = \sum_{j=0}^{k} h''_{0,j}(x)\, f_j + h''_{1,j}(x)\, f'_j + h''_{2,j}(x)\, f''_j\,, \tag{20c}$$

where

$$h''_{i,j}(x) = \gamma_{i,j}(x)\,\ell_j^3(x)$$

and

$$\gamma_{i,j}(x) = \phi''_{i,j}(x) + 6\phi'_{i,j}(x)\,\Sigma'\frac{1}{\left(x - x_i\right)} + 3\phi_{i,j}(x)\left\{3\left[\Sigma'\frac{1}{\left(x - x_i\right)}\right]^2 - \Sigma'\frac{1}{\left(x - x_i\right)^2}\right\},$$

where the $\phi_{i,j}(x)$ and $\phi'_{i,j}(x)$ are given in Equations 20a and 20b, and

$$\phi''_{0,j}(x) = 12\left[\ell_j'^{\,2}\left(x_j\right) - \frac{1}{4}\ell_j''\left(x_j\right)\right],$$

$$\phi''_{1,j}(x) = -6\ell_j'\left(x_j\right),$$

$$\phi''_{2,j}(x) = 1.$$

If the data values are equally spaced over the interval $\left[x_0, x_n\right]$, letting $h = x_i - x_{i-1}$ and $s = \left(x - x_0\right)/h$, the coefficients in Equations 18a, 19a, and 20a assume the following forms:

$$\ell_j(x) = \frac{(-1)^{n-j}\,\pi'(s - i)}{j!\,(n - j)!}\,, \tag{18a'}$$

$$\phi_{0,j}(x) = 1 - 2(s - j)\,\Sigma'\frac{1}{j - i}\,, \tag{19a'}$$

$$\phi_{1,j}(x) = h(s - j)\,,$$

$$\phi_{0,j}(x) = 1 + \frac{3}{2}(s-j)^2\left[3\left(\Sigma'\frac{1}{j-i}\right)^2 + \Sigma'\frac{1}{(j-i)^2}\right] - 3(s-j)\Sigma'\frac{1}{(j-i)}, \tag{20a'}$$

$$\phi_{1,j}(x) = h\left[(s-j) - 3(s-j)^2\,\Sigma'\frac{1}{(j-i)}\right],$$

$$\phi_{2,j}(x) = \frac{h^2}{2}(s-j)^2.$$

NUMERICAL EXAMPLES

In order to illustrate the power of the interpolation formulas including derivatives, Equations 18a', 19a', and 20' were applied to the following functions.

(a) The solution of the initial value problem:

$$\bar{y}'' = -\frac{\bar{y}}{\|\bar{y}\|^3}, \qquad \bar{y}(0) = \bar{y}_0, \qquad \bar{y}'(0) = \bar{y}_0',$$

where

$$\bar{y} = (y_1, y_2, y_3)$$

and

$$\|\bar{y}\| = (y_1^2 + y_2^2 + y_3^2)^{1/2}.$$

(b) The function $y = \sin(x)$.

In each case, the range of h was taken to be $1/2^m$ where $m = 0, 4(1)$ and $k = 1, 10(3)$. The calculations were performed on the UNIVAC 1107 computer using double-precision arithmetic. These results are given in Table 1.

CONCLUSION

The development of interpolating polynomials utilizing any number of successive derivatives of the function has been discussed. It has been shown that for any fixed N', such polynomials can be readily formulated. From the numerical results, we see that under the restrictive conditions of few data values and/or large stepsizes, accurate interpolation is possible by incorporating information concerning one or two derivatives. In general, it can be expected that for processes requiring accurate interpolation, if data concerning the derivatives are available or readily attainable, the polynomials discussed herein offer a distinct advantage of increased accuracy.

Table 1

Numerical Results.

m	k	Errors† for example (a) Eq. 18a'	Eq. 19a'	Eq. 20a'	Errors for example (b) Eq. 18a'	Eq. 19a'	Eq. 20a'
0	10	$.3 \times 10^{-4}$	$.4 \times 10^{-8}$	$.2 \times 10^{-12}$	$.2 \times 10^{-4}$	$.1 \times 10^{-13}$	$.4 \times 10^{-14}*$
	7	$.2 \times 10^{-3}$	$.3 \times 10^{-6}$	$.8 \times 10^{-9}$	$.2 \times 10^{-3}$	$.2 \times 10^{-10}$	$.9 \times 10^{-15}*$
	4	$.1 \times 10^{-2}$	$.3 \times 10^{-4}$	$.1 \times 10^{-6}$	$.5 \times 10^{-2}$	$.4 \times 10^{-6}$	$.8 \times 10^{-12}$
	1	$.7 \times 10^{-1}$	$.4 \times 10^{-2}$	$.1 \times 10^{-3}$	$.5 \times 10^{-1}$	$.1 \times 10^{-2}$	$.1 \times 10^{-4}$
1	10	$.2 \times 10^{-5}$	$.1 \times 10^{-9}$	$.5 \times 10^{-13}$	$.3 \times 10^{-7}$	$.7 \times 10^{-15}*$	$.9 \times 10^{-15}*$
	7	$.5 \times 10^{-4}$	$.1 \times 10^{-7}$	$.4 \times 10^{-11}$	$.3 \times 10^{-7}$	$.3 \times 10^{-14}$	$.3 \times 10^{-14}$
	4	$.3 \times 10^{-3}$	$.3 \times 10^{-6}$	$.6 \times 10^{-9}$	$.1 \times 10^{-3}$	$.4 \times 10^{-9}$	$.1 \times 10^{-14}$
	1	$.2 \times 10^{-1}$	$.2 \times 10^{-3}$	$.6 \times 10^{-5}$	$.7 \times 10^{-2}$	$.4 \times 10^{-4}$	$.8 \times 10^{-7}$
2	10	$.3 \times 10^{-6}$	$.5 \times 10^{-12}$	$.5 \times 10^{-14}*$	$.1 \times 10^{-10}$	$.5 \times 10^{-14}$	$.7 \times 10^{-14}$
	7	$.2 \times 10^{-5}$	$.1 \times 10^{-10}$	$.1 \times 10^{-13}$	$.1 \times 10^{-7}$	$.1 \times 10^{-14}$	$.1 \times 10^{-14}$
	4	$.3 \times 10^{-4}$	$.4 \times 10^{-7}$	$.2 \times 10^{-11}$	$.9 \times 10^{-5}$	$.2 \times 10^{-12}$	$.1 \times 10^{-14}$
	1	$.3 \times 10^{-2}$	$.2 \times 10^{-4}$	$.4 \times 10^{-7}$	$.9 \times 10^{-3}$	$.1 \times 10^{-5}$	$.6 \times 10^{-9}$
3	10	$.1 \times 10^{-9}$	$.8 \times 10^{-14}*$	$.1 \times 10^{-14}*$	$.1 \times 10^{-13}$	$.1 \times 10^{-14}$	$.2 \times 10^{-14}$
	7	$.3 \times 10^{-7}$	$.1 \times 10^{-13}$	$.2 \times 10^{-14}*$	$.2 \times 10^{-10}$	$.7 \times 10^{-15}*$	$.1 \times 10^{-14}$
	4	$.2 \times 10^{-5}$	$.5 \times 10^{-11}$	$.4 \times 10^{-14}*$	$.3 \times 10^{-6}$	$.1 \times 10^{-15}*$	$.4 \times 10^{-15}*$
	1	$.7 \times 10^{-3}$	$.7 \times 10^{-6}$	$.3 \times 10^{-9}$	$.1 \times 10^{-3}$	$.4 \times 10^{-7}$	$.5 \times 10^{-11}$
4	10	$.2 \times 10^{-11}$	$.1 \times 10^{-14}*$	$.6 \times 10^{-15}*$	$.4 \times 10^{-15}*$	$.9 \times 10^{-15}$	$.1 \times 10^{-14}$
	7	$.2 \times 10^{-9}$	$.3 \times 10^{-14}*$	$.2 \times 10^{-15}*$	$.5 \times 10^{-13}$	$.4 \times 10^{-15}*$	$.6 \times 10^{-15}*$
	4	$.3 \times 10^{-7}$	$.2 \times 10^{-13}$	$.7 \times 10^{-15}*$	$.1 \times 10^{-7}$	$.1 \times 10^{-15}*$	$.2 \times 10^{-15}*$
	1	$.4 \times 10^{-3}$	$.2 \times 10^{-7}$	$.5 \times 10^{-10}$	$.1 \times 10^{-4}$	$.1 \times 10^{-8}$	$.4 \times 10^{-13}$

† The error tabulated is the norm of the error vector.

* Indicates the error within the accuracy of the analytic solution.

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